

# Resolution of Identity Crisis of Events in Pile-up

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## Abstract

Mutually uncorrelated random discrete events, manifesting a common basic process, are examined often in terms of their occurrence rate as a function of one or more of their distinguishing attributes, such as measurements of photon spectrum as a function of energy. Such rate distributions obtained from the observed attribute values for an ensemble of events will correspond to the “true” distribution only if the event occurrence were *mutually exclusive*. However, due to finite resolution in such measurements, the problem of event *pile-up* is not only unavoidable, but also increases with event rate. Although extensive simulations to estimate the distortion due to pile-up in the observed rate distribution are available, no restoration procedure has yet been suggested. Here we present an elegant analytical solution to recover the underlying *true* distribution. Our method, based on Poisson statistics and Fourier transforms, is shown to perform as desired even when applied to distributions that are significantly distorted by pile-up. Our recipes for correction, as well as for prediction, of pile-up are expected to find ready applications in a wide variety of fields, ranging from high-energy physics to medical clinical diagnostics, and involving, but not limited to, measurements of count-rates and/or spectra of incident radiation using Charge Coupled Devices (CCDs) or other similar devices.

To formulate the *pile-up* problem analytically, let us denote the *true* and *apparent* event-rate distribution by  $\lambda(S)$  and  $\lambda_a(S)$  respectively, where  $S$  is a chosen attribute of the events we are interested in. Let  $\Delta$  be the (spatio-temporal) resolution with which these measurements are made. The basic situation in which ‘pile-up’ occurs can be illustrated using the following simple example. Consider two events (say, with  $S = S_1$  &  $S_2$ ) that occur within the measurement resolution  $\Delta$ . Since both the events occur within  $\Delta$  they would be mistaken for a single event with an attribute  $S_{sum} = S_1 + S_2$ . Therefore, instead of registering two events, one each at  $S = S_1$  and  $S = S_2$ , only one event at  $S = S_{sum}$  would be noted. As a result, the rate of events at  $S_1$  and at  $S_2$  are underestimated, with a corresponding overestimation of event rate at  $S_{sum}$ .

Generalizing the above example, if  $n$  events (say at,  $S_i$ ,  $i = 1 \rightarrow n$ ;  $n > 1$ ) were to overlap, then for each such occurrence, the event count at  $S_{sum} = \sum_{i=1}^n S_i$  is not only wrongly incremented by one, but a count is missed at each  $S_i$ . Consequently, in addition to this mistaken identity in ‘ $S$ ’, even the net event count suffers a deficit of  $n-1$ , for each unresolved occurrence of  $n$  events. Thus,

in general, any measured  $\lambda_a(S)$  deviates from the corresponding *true* distribution  $\lambda(S)$  due to a finite probability of unresolved events occurring within  $\Delta$ . It should be noted here that, even though the total rate of events is thus underestimated, the rate-weighted integral of  $S$  remains conserved, that is

$$\sum_i \lambda_a(S_i) \leq \sum_i \lambda(S_i) \quad (1)$$

$$\sum_i S_i \lambda_a(S_i) = \sum_i S_i \lambda(S_i) \quad (2)$$

Let us now consider events of a given fixed attribute value  $S_0$  and examine the probability of occurrence of one or more of such events. Since the mutually independent discrete events are expected to follow Poisson statistics, the probability of, in general,  $k$  ( $\geq 0$ ) such events occurring within the resolution  $\Delta$ , is given by the function

$$P_{poisson}(k; \lambda(S_0)) = \frac{[\lambda(S_0)]^k e^{-\lambda(S_0)}}{k!} \quad (3)$$

where  $\lambda$  is again the mean rate of events, or more specifically, the average number of events expected per  $\Delta$ . Thus  $k_i$  events, each with same attribute value  $S_i$ , occurring within  $\Delta$  would be unresolved,

and hence, they together would be mistaken for one event of strength  $S_i^a = k_i S_i$ . Thus, even though the true rate distribution is non-zero only at  $S = S_i$ , the *apparent* probability density distribution (PDD) spreads to all non-negative integral multiples of  $S_i$  when events are viewed with resolution  $\Delta$ , as

$$P_i^a(S_i^a = k_i S_i) = \frac{[\lambda(S_i)]^{k_i} e^{-\lambda(S_i)}}{k_i!} \quad (4)$$

with an implicit maximum event count of one per  $\Delta$ .

In general, for an ensemble of mutually independent discrete events with a range of the *true* attribute values (say,  $S_i$ ,  $i = 1 \rightarrow N$ ), the resultant PDD across the apparent attribute value  $S^a$  ( $= \sum_{i=1}^N S_i^a$ ) would be a grand convolution of the *apparent* PDDs (as in equation 4) corresponding to each of the respective apparent value  $S_i^a$ .

$$\begin{aligned} P^a(S^a = \sum_{i=1}^N k_i S_i) &= \bigotimes_{i=1}^N P_i^a(S_i^a = k_i S_i) \\ &= \bigotimes_{i=1}^N \frac{[\lambda(S_i)]^{k_i} e^{-\lambda(S_i)}}{k_i!} \end{aligned} \quad (5)$$

where  $\bigotimes$  denotes convolution product, and again,  $k_i$  is the number of events with *true* attribute  $S_i$ , occurring within  $\Delta$ .

If  $\tilde{P}^a(f)$  &  $\tilde{P}_i^a(f)$  are the Fourier transforms of  $P^a(S^a)$  &  $P_i^a(S_i^a)$  respectively, the *convolution theorem* would relate them as follows,

$$\tilde{P}^a(f) = \prod_{i=1}^N \tilde{P}_i^a(f) \quad (6)$$

where  $\prod$  denotes a simple product.

The Fourier transforms appearing in the product on the right-hand side of the above equation, *i.e.*  $\tilde{P}_i^a(f)$ , can be obtained in general for any  $i$ , by summing over  $k_i$  the Fourier contribution from each of the components of  $P_i^a(S_i^a)$  (see equation 4), evaluated at the discrete values of  $S_i^a = k_i S_i$ .

Thus,

$$\begin{aligned} \tilde{P}_i^a(f) &= \sum_{k_i=0}^{\infty} \left( \frac{[\lambda(S_i)]^{k_i} e^{-\lambda(S_i)}}{k_i!} \right) e^{-j2\pi k_i S_i f} \\ &= e^{-\lambda(S_i)} \sum_{k_i=0}^{\infty} \frac{[\lambda(S_i) e^{-j2\pi S_i f}]^{k_i}}{k_i!} \\ &= e^{-\lambda(S_i)} e^{[\lambda(S_i) e^{-j2\pi S_i f}]} \end{aligned} \quad (7)$$

Substituting this result in equation 6, we get

$$\begin{aligned} \tilde{P}^a(f) &= \prod_{i=1}^N e^{-\lambda(S_i)} e^{[\lambda(S_i) e^{-j2\pi S_i f}]} \\ &= e^{-\sum_{i=1}^N \lambda(S_i)} e^{\sum_{i=1}^N \lambda(S_i) e^{-j2\pi S_i f}} \\ &= e^{-\sum_{i=1}^N \lambda(S_i)} e^{\tilde{P}(f)} \end{aligned} \quad (8)$$

where  $\tilde{P}(f) = \sum_{i=1}^N \lambda(S_i) e^{-j2\pi S_i f}$ , which is the Fourier transform of the *true* distribution.

By taking natural logarithm of both sides and rearranging, we obtain

$$\ln(\tilde{P}^a(f)) = \tilde{P}(f) - \sum_{i=1}^N \lambda(S_i) \quad (9)$$

This relation between the Fourier transforms of the *true* and the *apparent* distributions of events, should enable recovery of the underlying *true* rates or counts of events, as a function of a chosen attribute, from the corresponding *observed* distribution, often distorted due to pile-up.

The suggested recipe is

1. One begins with measurements over a total number of, say,  $M$  independent resolution cells, each of size  $\Delta$ , providing a record of discrete events (numbering, say,  $N_c$ , where  $N_c \leq M$ ). Using such data, the events are sorted and counted according to the apparent value of their chosen attribute ( $S^a$ ). The sorted event-count distribution, say  $C^a(S^a)$ , is normalized

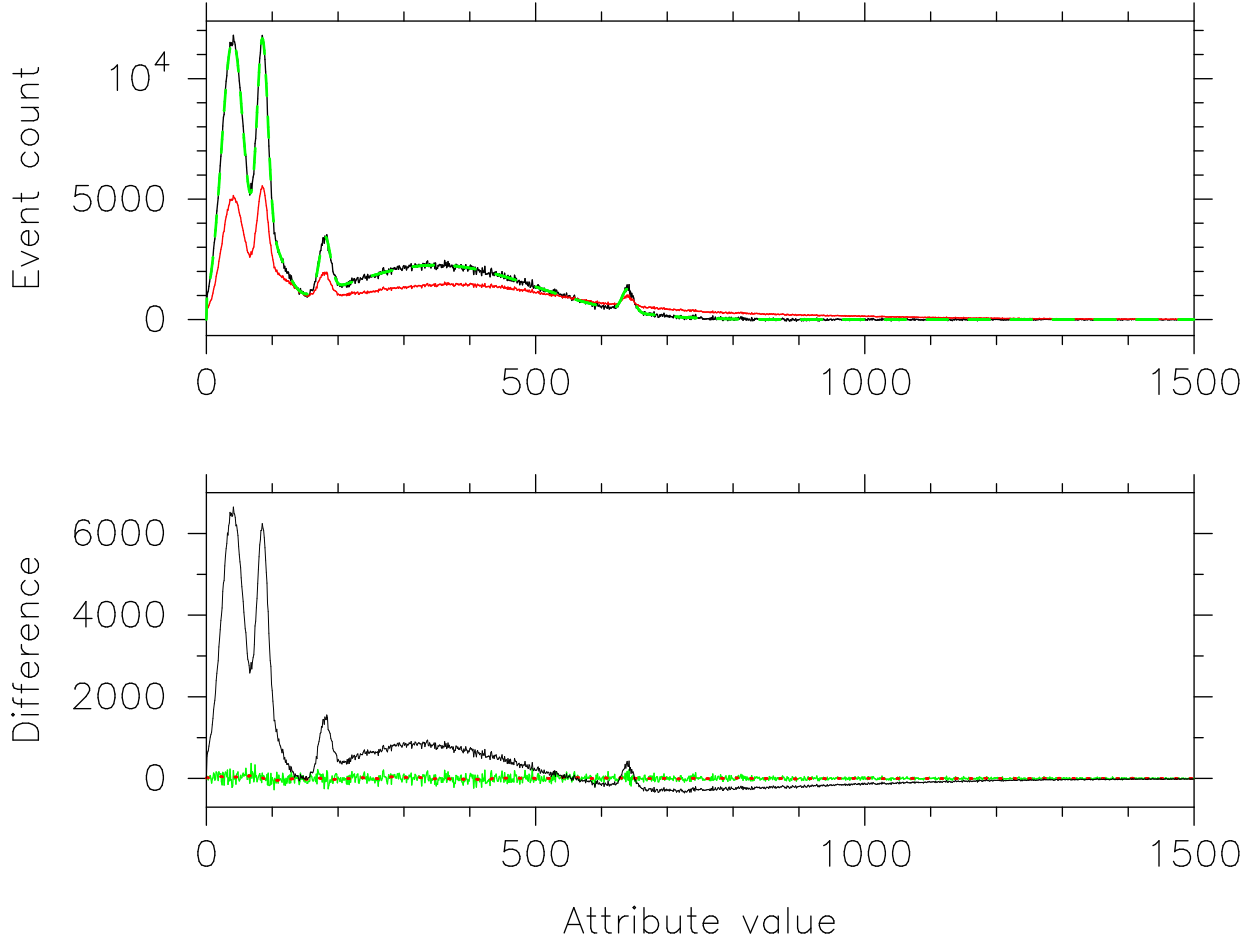


Figure 1: Illustration of a distribution affected by pile-up (top panel; red) which is obtained through Monte-Carlo simulations, from an assumed model for a “true” distribution (top panel; green). The resultant distribution after pile-up correction (top panel; black) matches the model “true” distribution, within the statistical uncertainties. The bottom panel shows the amount of estimated correction (bottom panel; black), the difference with the recovered and the model distributions (bottom panel; green), along with a smoothened version of the latter (bottom panel; red).

by  $M$  to obtain an apparent probability density distribution (PDD) of events across  $S^a$ , that is  $P^a(S^a) = C^a(S^a)/M$ . The PDD value at  $S^a = 0$ , if not known a priori or explicitly, can be estimated trivially and is  $\geq 0$ , such that the total probability (including that at  $S^a = 0$ ) equals unity.

2.  $P^a(S^a)$  is Fourier transformed to obtain a so-called characteristic function,  $\tilde{P}^a(f)$ , but avoiding any normalization by the number of points ( $N$ ) transformed.
3. This crucial step involves computing  $\tilde{X}(f) =$

$\ln(\tilde{P}^a(f))$ , such that if  $\tilde{P}^a(f) = a(f)e^{j\phi(f)}$ , then  $\tilde{X}(f) = \ln(a(f)) + j\phi(f)$ , where  $j = \sqrt{-1}$ .

4. Inverse Fourier transforming  $\tilde{X}(f)$  (now with usual normalization by  $N$ ) gives the *true* event-rate distribution  $\lambda(S)$  across the  $S$  range, along with a *dip* at  $S = 0$  whose magnitude is  $\sum_{i=1}^N \lambda(S_i)$  for  $S_i \neq 0$ .
5. At  $S = 0$ , one may ignore this (dip) contribution completely, or compare its magnitude with the integral over the rest of the  $S$ -range to assess internal consistency. The  $\lambda(S)$  thus

obtained is multiplied by  $M$  to get the estimate of true distribution of event counts  $C(S)$ , or further divided by  $\Delta$  to get the underlying event-rate distribution in relevant basic units (such as per unit area and/or per unit time).

It is important to satisfy the constraint, integral of  $P^a(S^a)$  being equal to unity, in general, and also to ensure that the rate-weighted integral of  $S$  is conserved as desired (see Equation 2) through the above restoration procedure. It is easy to show from Equation 9 that

$$\left(\frac{d\tilde{P}^a}{df}\right)_{f=0} = \tilde{P}^a(f=0) \left(\frac{d\tilde{P}}{df}\right)_{f=0} \quad (10)$$

where the first derivatives of  $\tilde{P}$  and  $\tilde{P}^a$  with respect to  $f$ , when evaluated at  $f = 0$ , correspond to respective rate-weighted integrals of  $S$ , or to the first moments of the respective distributions. In contrast with the above, the true event-rate distribution  $\lambda(S)$  is not expected to follow any such constraint on its integral.

Figure 1 illustrates application of our procedure, and its result. For simplicity, and without loss of generality, all distributions have been binned with  $S$ -interval of unity. The level of correction effected by the procedure is clearly evident on the left-side part of the distribution, where this *homecoming* of counts is accompanied by corresponding *deportation* out of the right-side region. In the present example, the restored count totals to about 1.7 million (consistent with the original model distribution), of which about 32% was lost due to pile-up. Although the original distribution is confined to attribute values  $S < 900$ , the apparent distribution extended well beyond due to pile-up, with about 45000 counts in the range  $S > 900$ . We note that the restored distribution is seen also confined to  $S < 900$ , and at larger  $S$ , any deviation now from the expected count (i.e. zero) is found to be well within statistical uncertainties.

It is important to emphasize that the above procedure for pile-up correction works equally well also for two-sided distributions. Note that when the distribution  $P^a(S^a)$  is one-sided (either  $S \geq 0$ , as in Figure 1, or  $S \leq 0$ ), the real and imaginary

parts of  $\tilde{P}^a(f)$  represent a Hilbert pair, and so do the corresponding parts of  $\ln(\tilde{P}^a(f))$ , consistent with the recovered  $P(S)$  also being one-sided. Also, the derived relation (Eq. 9) provides a direct way for predicting a piled-up distribution, if the *true* distribution is known.

## Applications and Discussion

Pile-up effects have been encountered and discussed in a wide variety of contexts and measurements over the past several decades, as apparent from the non-exhaustive list below.

1. In X-ray astronomy (e.g. the Suzaku [16] and the Chandra [4] missions), particularly where CCDs or similar detectors are employed for measuring (photon) energy spectra, apart from imaging, of celestial sources.
2. In high-energy physics experiments[6] (e.g. using the Large Hadron Collider at CERN, including the on-going hunt for Higgs boson[3]), particle detectors are equipped with triggers to evaluate interaction among high energy particles which are being studied. Cosmic ray detectors are also equipped with similar triggers which are enabled when cosmic rays of sufficiently high energy enter the Cherenkov detectors [7]. In both contexts, *fake* triggers can occur due to pile-up of multiple events of lower energy, and on the other hand, using higher thresholds (to reduce such *ghost* or *phantom* particles) can “miss” to detect a real particle.
3. In radiation measurement application and/or in Gamma spectroscopy etc., using solid state detectors (e.g. Si(Li); NaI(Tl))[5, 10, 11]
4. In medical clinical diagnostics [1, 8, 15] such as radio-nuclide therapy dosimetry imaging, micro-dosimetry of inhaled  $\alpha$ -emitters (e.g. in measurements of specific energy spectra of epithelial cells of bronchiolar airways) and cardiac first-pass imaging, using Gamma Cam-

5. In neutrino mass determination, using micro-calorimeter to measure the entire spectrum of  $^{187}\text{Re}$  (MARE experiment [13]).

The prevailing approaches to tackle the pile-up issue are: a) use as high a spatio-temporal resolution as possible, b) reduce rate of events, if controllable, or c) restrict to regions with significantly reduced event rates, thus ignoring potentially valuable data from regions that would be rich in events. The latter two result in poor statistics, compromising sensitivity of measurements. In the absence of any correction procedure so far, iterative procedures[5, 13, 10, 11] to seek the underlying “true” distribution are in use, wherein Monte-Carlo simulations of events, following assumed models of the “true” distribution and of pile-up, are employed to obtain simulated apparent distributions that are compared with those measured.

The simple relation derived by us (Eq. 9) and the correction procedure presented here should find ready applicability, in the above mentioned and other relevant areas, for recovering the underlying *true* distribution of events, even when the observed distributions have significant distortion due to pile-up. This, in turn, would enable such measurements with significantly improved sensitivity, and detection of features or events otherwise masked by the pile-up distortion.

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